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Uniform Approximation by Generalized Splines with Free Knots

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1. INTRODUCTION

Best approximation by polynomial and Tchebycheffian spline functions with both fixed and free knots has been thoroughly investigated in a number of papers (see, e.g., [5] and references therein). Recently, in [8], we have examined best approximation by certain classes of generalized splines with fixed knots. The purpose of this paper is to study best approximation by similar classes of generalized splines, but with *free knots*. Our main results concern *segment approximation* and approximation by *continuously composed Tchebycheff systems*. For these spaces, we treat the

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usual best approximation questions of existence, uniqueness, and characterization.

We begin by defining the spline spaces of interest. Suppose $\mathscr{U} = \{u_1, ..., u_m\}$ is a set of m linearly independent functions in $C[a, b]$. For most of the results of this paper we shall assume that $\mathscr{U}$ is a Tchebycheff system, and at times will restrict $\mathscr{U}$ even further.

Given a partition Δ of $[a, b]$ defined by $a = x_0 < \cdots < x_{r+1} = b$, let $I_i = [x_i, x_{i+1})$ and $J_i = [x_i, x_{i+1}]$ for $i = 0, ..., r-1$. Let $J_r = I_r = [x_r, x_{r+1}]$. We define the space of *piecewise $\mathscr{U}$-polynomials with knots at Δ as*

$$\mathscr{P}\mathscr{U}(\Delta) = \{s: [a, b] \to \mathbb{R}: s|_{I_i} \in \mathscr{U},\ i = 0, ..., r\}. \tag{1.1}$$

Given a positive integer k, we define the space of *piecewise $\mathscr{U}$-polynomials with k free knots* as

$$\mathscr{P}\mathscr{U}_k = \bigcup \{\mathscr{P}\mathscr{U}(\Delta): \#(\Delta) \leqslant k\}. \tag{1.2}$$

While $\mathscr{P}\mathscr{U}(\Delta)$ is a linear space, clearly $\mathscr{P}\mathscr{U}_k$ is a nonconvex set. We shall also be interested in the following spaces of smoother piecewise functions, provided that the elements of $\mathscr{U}$ are in $C^l[a, b]$:

$$\mathscr{S}\mathscr{U}^l(\Delta) = \mathscr{P}\mathscr{U}(\Delta) \cap C^l[a, b] \tag{1.3}$$

$$\mathscr{S}\mathscr{U}^l_k = \mathscr{P}\mathscr{U}_k \cap C^l[a, b]. \tag{1.4}$$

Approximation using $\mathscr{P}\mathscr{U}_k$ is the topic of Section 2, where we discuss general functions in $C[a, b]$ as well as functions belonging to the convexity cone associated with $\mathscr{U}$. Results on approximation using $\mathscr{S}\mathscr{U}^0_k$ can be found in Section 3, while in Section 4 we show that for $l \geqslant 1$, the spaces $\mathscr{S}\mathscr{U}^l_k$ are not suited for approximation since there exists a function in $C^{l-1}[a, b]$ which has no best approximation in $\mathscr{S}\mathscr{U}^l_k$. In Section 5 we use the methods of this paper to give an improved necessary condition for approximation by *polynomial splines* with free knots.

In the remainder of this section we introduce some notation. Throughout this paper we shall be concerned with the uniform norm. Given a function g and interval J, we denote the uniform norm of g on the interval J by $\|g\|_J$. When there is no confusion about which interval we are working with, we suppress the subscript J. Given a space of functions S and a function $f \in C[a, b]$, we denote the *distance* of f to S by

$$d(f, S) = \inf_{s \in S} \|f - s\|.$$

If $g \in C[c, d]$, we say that g *alternates* p times on the closed interval $J = [c, d]$ provided that there exist points $c \leqslant t_0 < \cdots < t_p \leqslant d$ and $\sigma \in \{-1, 1\}$ such that

$$g(t_j) = (-1)^j \sigma \|g\|_{[c,d]}, \qquad j = 0, ..., p. \tag{1.5}$$

The points $\{t_0, ..., t_p\}$ are called *alternating extreme points* of g. In general, we write

$$A_J(g) = \max\{p : g \text{ has } p+1 \text{ alternating extreme points in } J\}.$$

It will also be useful to have a notation for the pieces of splines. Given $s \in \mathscr{P}\mathscr{U}_k$ with $r \leqslant k$ knots

$$a = x_0 < \cdots < x_{r+1} = b, \tag{1.6}$$

we define

$$s_i(x) = \begin{cases} s(x), & \text{for } \quad x_i \leqslant x < x_{i+1} \\ \lim_{t \to x_{i+1}} s(t), & \text{for } \quad x = x_{i+1}, \end{cases} \tag{1.7}$$

for $i = 0, ..., r$.

2. Segment Approximation

In this section we are interested in approximating a given function $f \in C[a, b]$ by functions in $\mathscr{P}\mathscr{U}_k$. To set the stage, we begin by stating a well-known general result on segment approximation (cf. [3, 4]).

THEOREM 2.1. *Suppose $\mathscr{U}$ is an arbitrary set of m functions in $C[a, b]$. Then for any $f \in C[a, b]$, there exists at least one best approximation of f in $\mathscr{P}\mathscr{U}_k$. Let $s \in \mathscr{P}\mathscr{U}_k$ have r knots as in* (1.6) *such that*

$$\|f - s_i\|_{J_i} = d(f, \mathscr{P}\mathscr{U}_k)_{J_i} = \|f - s\|_{[a,b]}, \qquad i = 0, ..., r. \tag{2.1}$$

Then s is a best approximation of f in $\mathscr{P}\mathscr{U}_k$. While not every best approximation necessarily satisfies (2.1), *there exists at least one which does. Moreover, there exists at least one best approximation s of f such that $|f - s| \in C[a, b]$.*

In general, a given $f \in C[a, b]$ will have more than one best approximation, and it is possible that none of the best approximations belong to $C[a, b]$ (see Example 2.2 below). Theorem 2.1 asserts that s is a best approximation if for each i, the piece s_i defined in (1.7) is a best approximation of f on J_i, and, in addition, that the errors $d_i = \|f - s_i\|_{J_i}$, $i = 1, ..., n$, are *balanced*. Example 2.2 also shows that for s to be a best approximation, it is not sufficient that the errors d_i be balanced.

EXAMPLE 2.2. Let f be the function

$$f(x)=\begin{cases} x, & 0\leqslant x\leqslant 1\\ 2-x, & 1\leqslant x\leqslant 2\\ 0, & 2\leqslant x\leqslant 3\\ 3-x, & 3\leqslant x\leqslant 4\\ x-5, & 4\leqslant x\leqslant 5,\end{cases}$$

and consider approximation from $\mathscr{P}\mathscr{U}_1$ on $[0, 5]$ where $\mathscr{U}$ consists of $u_1=1$.

Discussion. For each $2\leqslant \xi\leqslant 3$, let

$$s_{\xi,h}=\begin{cases} h, & 0\leqslant x\leqslant \xi\\ -h, & \xi<x\leqslant 5.\end{cases}$$

These splines all have balanced errors, but they are best approximations of f from $\mathscr{P}\mathscr{U}_1$ only on the case when $h=0.5$. This function has no continuous best approximation. ▌

If $\mathscr{U}$ is a Tchebycheff system, then the sufficient condition of Theorem 2.1 can be restated in terms of certain alternation conditions.

THEOREM 2.3. *Suppose $\mathscr{U}=\{u_i\}_1^m$ is a Tchebycheff system and that $f\in C[a,b]$. Let $s\in\mathscr{P}\mathscr{U}_k$ be such that s has r knots as in* (1.6). *Suppose that the errors are balanced as in* (2.1), *and that*

$$A_{J_i}(f-s_i)\geqslant m, \qquad i=0, ..., r. \tag{2.2}$$

Then s is a best approximation of f in $\mathscr{P}\mathscr{U}_k$. Moreover, there exists at least one best approximation such that (2.2) *holds.*

Theorem 2.3 gives only a sufficient condition for a function $s\in\mathscr{P}\mathscr{U}_k$ to be a best approximation of a given function f. In order to get a complete characterization, we must put some conditions on the function f. In the remainder of this section, we restrict ourselved to functions in the convexity cone associated with $\mathscr{U}$. Assuming that $\mathscr{U}$ is a Tchebycheff system satisfying

$$D\begin{pmatrix} t_1, ..., t_m\\ u_1, ..., u_m\end{pmatrix}=\det(u_j(t_i))_{i,j=1}^m>0 \qquad \text{for all} \quad \alpha\leqslant t_1<t_2<\cdots<t_m\leqslant b,$$

the corresponding *convexity cone* is defined by

$$\mathscr{K}(\mathscr{U})=\{f\in C[a,b]: D\begin{pmatrix} t_1, ..., t_{m+1}\\ u_1, ..., u_m, f\end{pmatrix}>0$$
$$\text{for all } a\leqslant t_1<t_2<\cdots<t_{m+1}\leqslant b\}. \tag{2.3}$$

Thus, if $f \in \mathcal{K}(\mathcal{U})$, then the set $\{u_1, ..., u_m, f\}$ forms an $m+1$ dimensional Tchebycheff system.

In Theorem 2.7 below we give a complete characterization of best approximations from $\mathcal{P}\mathcal{U}_k$ under the assumption that $f \in \mathcal{K}(\mathcal{U})$. First we need a lemma.

LEMMA 2.4. *Suppose that $\mathcal{U}$ is a* normed Tchebycheff system; *i.e.*, $u_1 = 1$. *Let $f \in \mathcal{K}(\mathcal{U})$, and suppose that $[c, d]$ is a proper subset of $[a, b]$. Let u be a best approximation of f from $\mathcal{U}$ on $[c, d]$. Then $f - u$ alternates exactly m times on $[c, d]$. More precisely, there exists points $c = t_0 < t_1 < \cdots < t_m = d$ and $\sigma \in \{-1, 1\}$ such that*

$$(-1)^i \sigma(f-u)(t_i) = \|f-u\|_{[c,d]}, \qquad i = 0, ..., m. \tag{2.4}$$

In particular, both c and d are peak points. Moreover,

$$\text{for all} \quad a \leqslant \alpha < c, \qquad \|f-u\|_{[c,d]} < \|f-u\|_{[\alpha,d]}, \tag{2.5}$$

$$\text{for all} \quad d < \beta \leqslant b, \qquad \|f-u\|_{[c,d]} < \|f-u\|_{[c,\beta]}. \tag{2.6}$$

Proof. Since $\mathcal{U}$ is a Tchebycheff system, $A_{[c,d]}(f-u) \geqslant m$. The function $f-u$ cannot have more than m alternations, for if it did, then it would have at least $m+1$ zeros. This is impossible since $f \in \mathcal{K}(\mathcal{U})$ implies that $f-u$ also lies in $\mathcal{K}(\mathcal{U})$. We now prove (2.5). Suppose that $a \leqslant \alpha < c$, and that $\|f-u\|_{[c,d]} = \|f-u\|_{[\alpha,d]}$. Then at least one of the functions $g = f - u - \|f-u\|_{[\alpha,d]}$ or $h = f - u + \|f-u\|_{[\alpha,d]}$ has at least $m+1$ zeros on $[\alpha, d]$, where we count each interior double zero twice. But this is impossible since $1 \in \mathcal{U}$ implies that both g and h lie in the $m+1$ dimensional Tchebycheff space spanned by $\{u_1, ..., u_m, f\}$, and hence can have at most m zeros (cf., Theorem 4.2 of [2]). Clearly (2.5) implies that $|(f-u)(c)| = \|f-u\|_{[c,d]}$; i.e., c is a peak point of $f-u$. A similar analysis establishes (2.6) and the fact that d must also be a peak point of $f-u$. ∎

We now give examples to show that Lemma 2.4 fails if we drop either the assumption that $\mathcal{U}$ contains $u_1 \equiv 1$, or that $f \in \mathcal{K}(\mathcal{U})$.

EXAMPLE 2.5. Let

$$f(x) = \begin{cases} -1, & 0 \leqslant x \leqslant 1 \\ x-2, & 1 \leqslant x \leqslant 3 \\ 1, & 3 \leqslant x \leqslant 4, \end{cases}$$

and consider approximation from the Tchebycheff system $\mathcal{U}$ on $[0, 4]$ consisting of

$$u_1(x) = \begin{cases} x+1, & 0 \leqslant x \leqslant 2 \\ 5-x, & 2 \leqslant x \leqslant 4. \end{cases}$$

Discussion. It is easy to check that f is in $\mathscr{K}(\mathscr{U})$. Now for all $0 \leqslant c \leqslant 1$ and $3 \leqslant d \leqslant 4$, the best approximation of f on $[c, d]$ is $u=0$, but (2.5), (2.6) fail to hold.

EXAMPLE 2.6. Let f be the function in Example 2.5, and consider approximation from the Tchebycheff system $\mathscr{U}$ consisting of the function $u_1 = 1$ on $[0, 4]$.

Discussion. Since $f - u_1 = 0$ on $[3, 4]$, f is not in $\mathscr{K}(\mathscr{U})$. Now for all $0 \leqslant c \leqslant 1$ and $3 \leqslant d \leqslant 4$, the best approximation of f on $[c, d]$ is $u=0$, but (2.5), (2.6) fail to hold.

We are now ready for our characterization theorem.

THEOREM 2.7. *Suppose $\mathscr{U}$ is a normed Tchebycheff system, and that $f \in \mathscr{K}(\mathscr{U})$. Then there exists a unique best approximation s of f in $\mathscr{P}\mathscr{U}_k$. Moreover, $s \in \mathscr{P}\mathscr{U}_k$ is the unique best approximation if and only if it has knots $a = x_0 < x_1 < \cdots < x_{k+1} = b$ such that*

$$A_{J_i}(f - s) = m, \qquad i = 0, ..., k,$$

and the errors are balanced with

$$\|f - s_i\|_{J_i} = d := d(f, \mathscr{P}\mathscr{U}_k), \qquad i = 0, ..., k.$$

Proof. The existence of a best approximation follows from Theorem 2.1. Theorem 2.3 asserts that there exists a best approximation s of f such that $f - s$ alternates at least m times on each subinterval J_i defined by the knots, and the errors are balanced. Since $f \in \mathscr{K}(\mathscr{U})$, the function $f - s$ cannot alternate more than m times on any interval.

We now prove that if s is a best approximation, then it must possess k knots. Suppose that s has only $r < k$ knots, $a = x_0 < x_1 < \cdots < x_{r+1} = b$. Let $\tilde{x}_i = x_i$ for $i = 0, ..., r$, and let $\tilde{x}_{r+1} = (x_r + x_{r+1})/2$ and $\tilde{x}_{r+2} = b$. Now by Lemma 2.4,

$$d(f, \mathscr{U})_{[\tilde{x}_{r+1}, b]} < d(f, \mathscr{U})_{[\tilde{x}_r, b]}.$$

Let $\tilde{s}_0, ..., \tilde{s}_{r+1}$ be the best approximations of f on the intervals $\tilde{J}_0, ..., \tilde{J}_{r+1}$. We have

$$\|f - \tilde{s}\|_{\tilde{J}_{r+1}} < d,$$

where $\tilde{s}$ is the function whose pieces are $s_0, ..., \tilde{s}_{r+1}$. Now it is clear that starting with $\tilde{x}_{r+1}$, we can move each of the knots of $\tilde{s}$ leftward to obtain a better approximation of f. This contradiction establishes our assertion that s must have k knots.

It remains to prove the uniqueness of s. If $\tilde{s}$ is another best approximation of f from $\mathscr{P}\mathscr{U}_k$, it must also have k knots, say $a=\tilde{x}_0<\tilde{x}_1<\cdots<\tilde{x}_{k+1}=b$. But then for some i, j, the set $J_i=[x_i, x_{i+1}]$ is a proper subset of $\tilde{J}_j=[\tilde{x}_j, \tilde{x}_{j+1}]$. But then by Lemma 2.4,

$$d=\|f-s\|_{J_i}<\|f-\tilde{s}\|_{\tilde{J}_j}\leqslant d.$$

From this contradiction we conclude that s is the unique best approximation. ∎

Theorem 2.7 fails if any of the hypotheses are not satisfied. The following example shows what happens when f is not in $\mathscr{K}(\mathscr{U})$.

EXAMPLE 2.8. Let f be the function in Example 2.2, and consider approximation from the space $\mathscr{P}\mathscr{U}_2$ with $\mathscr{U}$ the set consisting of the single function $u_1=1$.

Discussion. It is not hard to see that $d(f, \mathscr{P}\mathscr{U}_2)=0.5$, and that this distance can be achieved only if the knots are chosen with $x_1\geqslant 1.5$ and $x_2\leqslant 3.5$. Now clearly there are many best approximations, some of which have only one knot. Clearly, all best approximations are discontinuous.

We conclude this section by showing that under the hypotheses of Theorem 2.7, if m is even, the best approximation is continuous.

THEOREM 2.9. *Suppose $\mathscr{U}$ is a normed Tchebycheff system consisting of m functions with m even, and suppose that $f\in\mathscr{K}(\mathscr{U})$. Then the unique best approximation of f from $\mathscr{P}\mathscr{U}_k$ belongs to $C[a, b]$.*

Proof. Let s be the unique best approximation of f, and suppose that its knots are given by $a=x_0<x_1<\cdots<x_{k+1}=b$. It is clear that s is continuous at all points except possibly the x_i. Thus, it suffices to show that for each $i=1, ..., k$, we have $s_{i+1}(x_i)=s_i(x_i)$. Fix $1\leqslant i\leqslant k$. Then since $f-s$ alternates exactly m times on J_i, it follows that $f-s$ must have zeros at some points $x_i<z_{i1}<\cdots<z_{im}<x_{i+1}$. Then, it is easy to see that for all $x\in J_i$,

$$(f-s_i)(x)=D\begin{pmatrix} z_{i1}, ..., z_{im}, x \\ u_1, ..., u_m, f \end{pmatrix}\Big/ D\begin{pmatrix} z_{i1}, ..., z_{im} \\ u_1, ..., u_m \end{pmatrix}.$$

Since m is even while each of the knots is a peak point of $f-s$, this implies that $(f-s_i)(x_i)=(f-s_i)(x_{i+1})=d$. The result follows. ∎

Example 2.8 shows that Theorem 2.9 fails when f is not in $\mathscr{K}(\mathscr{U})$. The following example shows that it also fails when m is odd.

EXAMPLE 2.10. Let $f(x)=x$ on the interval $[-1, 1]$, and consider approximation from $\mathscr{P}\mathscr{U}_1$ with $\mathscr{U}$ the set consisting of $u_1\equiv 1$.

Discussion. All of the hypotheses of Theorem 2.9 hold except that m is odd. Clearly, the unique best approximation of f is the discontinuous spline $s = -0.5 + x_+^0$.

3. Approximation from $\mathcal{S}\mathcal{U}_k^0$

In this section we are interested in approximating a given function $f \in C[a, b]$ by functions in the space $\mathcal{S}\mathcal{U}_k^0$ defined in (1.4). Since $\mathcal{S}\mathcal{U}_k^0 \equiv \mathcal{U}$ in the case $m = 1$, we shall restrict our attention to the case where $m \geqslant 2$. We begin with an existence result.

THEOREM 3.1. *Given any $f \in C[a, b]$, there exists at least one best approximation of f from the space $\mathcal{S}\mathcal{U}_k^0$.*

Proof. Let $s_n \in \mathcal{S}\mathcal{U}_k^0$ be a sequence with

$$\|f - s_n\| \to d := d(f, \mathcal{S}\mathcal{U}_k^0).$$

Let $a = x_{n,0} < \cdots < x_{n,k_n+1} = b$ be the knots of s_n. Since each s_n has at most k knots, there is a subsequence, which we denote by s_n again, such that $s_n \to s$ with $s \in \mathcal{P}\mathcal{U}_k$ and $\|f - s\| = d$. If $s \in C[a, b]$, we are done, so we suppose now that s is discontinuous at some knot.

Suppose the knots of s are $a = x_0 < x_1 < \cdots < x_{r+1} = b$. Since each s_n is continuous, if s has a jump at some knot x_j, then there must be a pair of knots $x_{n,i}$ and $x_{n,i+1}$ both of which converge to x_j. It follows that if s has p jump discontinuities, then $r \leqslant k - p$. We now show how to modify s to construct a function $\tilde{s} \in \mathcal{S}\mathcal{U}_k^0$ with $\|f - \tilde{s}\| = d$.

Suppose $p = 1$, and let x_j be the knot where s has a jump discontinuity. Without loss of generality, we may assume that $s(x_j^+) - s(x_j^-) > 0$, and that for some sufficiently small $\delta > 0$, $f(x) - s(x) > 0$ in $(x_j - \delta, x_j)$. Assume that $s_R \in \mathcal{P}\mathcal{U}_k$ is such that $s_R \equiv s$ on $[x_j, b]$ and that for all $x \in [a, x_j)$, s_R is defined to be the same polynomial as on the interval beginning at x_j. Similarly, let $s_L \in \mathcal{P}\mathcal{U}_k$ be such that $s_L \equiv s$ on the $[a, x_j)$, and that for all $x \in [x_j, b]$, s_L is defined to be the same polynomial as on the interval ending at x_j. Since $\mathcal{U}$ is a Tchebycheff system on $[a, b]$ and thus a complete Tchebycheff system on (a, b), the functions $u_1, ..., u_i$ form a Tchebycheff system on (a, b) for each $i = 1, ..., m$ (see [10]). We now divide the proof into two cases.

Case 1. $(m = 2)$. In this case there exists a unique $q \in \operatorname{span}\{u_1, u_2\}$ such that

$$g(x_j - \delta) = s_L(x_j - \delta) \qquad \text{and} \qquad g(x_j) = s_R(x_j).$$

Let

$$\tilde{s}(x)=\begin{cases} g(x), & x \in (x_j-\delta, x_j) \\ s(x), & \text{otherwise.} \end{cases}$$

This construction has removed the jump discontinuity of s at x_j. Moreover, since s_L-g and s_R-g have no further zeros in (a, b), it follows that $s_L(x)<\tilde{s}(x)<s_R(x)$, if $x \in (x_j-\delta, x_j)$. Hence $\|f-\tilde{s}\| \leqslant \|f-s\|=d$.

Case 2. $(m \geqslant 3)$. In this case, there exists a unique $g \in \operatorname{span}\{u_1, u_2, u_3\}$ such that $g(x_j-\delta)=s_L(x_j-\delta)$, $g(x_j)=(s_L(x_j)+s_R(x_j))/2$, and $g(x_j+\delta)=s_R(x_j+\delta)$. Assume that $\tilde{x}$ is the biggest zero of s_L-g in $(x_j-\delta, x_j)$ and $\hat{x}$ is the smallest zero of s_R-g in $(x_j, x_j+\delta)$. Let

$$\tilde{s}(x)=\begin{cases} g(x), & x \in (\tilde{x}, \hat{x}) \\ s(x), & \text{otherwise.} \end{cases}$$

This construction has removed the jump discontinuity of s at x_j. Moreover, since $s_L(x_j)<g(x_j)<s_R(x_j)$, and by the choice of $\tilde{x}$ and $\hat{x}$, it follows that $s_L(x)<\tilde{s}(x)<s_R(x)$, if $x \in (\tilde{x}, \hat{x})$. Hence, $\|f-\tilde{s}\| \leqslant \|f-s\|=d$.

The above construction was carried out for the case where $p=1$, i.e., where there is just one knot where s has a jump. If $p>1$ we may apply the same construction at each of the p knots. Since as noted above, $r \leqslant k-p$, the resulting spline $\tilde{s}$ still has at most k knots. Since $\|f-\tilde{s}\| \leqslant d$, it follows that $\tilde{s}$ is a best approximation of f from $\mathcal{S}\mathcal{U}_k^0$. ∎

Theorem 3.1 shows the existence of best approximations of functions $f \in C[a, b]$ by splines in the space $\mathcal{S}\mathcal{U}_k^0$. In general, we cannot expect uniqueness (examples are easy to construct). Our next step is to try to characterize best approximations in terms of alternations. The following result provides a sufficient condition for best approximations, under an additional assumption on the space $\mathcal{U}$.

THEOREM 3.2. *Suppose that $\mathcal{U}=\{u_1, ..., u_m\}$ is a normed Tchebycheff system of functions in $C^1[a, b]$. Suppose that $\mathcal{U}$ has the additional property that*

$$\mathcal{U}' := \{u_2', ..., u_m'\} \tag{3.1}$$

is also a Tchebycheff system on $[a, b]$. Given $f \in C[a, b]$, suppose that s is a spline in $\mathcal{S}\mathcal{U}_k^0$ with knots $a=x_0<x_1< \cdots <x_{r+1}=b$. In addition, suppose that there exist p, q and points $x_p \leqslant t_0< \cdots <t_N \leqslant x_{p+q+1}$ with $N:=(q+1)m+k-q-l$ such that

$$f(t_i)-s(t_i)=(-1)^i \sigma d, \qquad i=0, ..., N,$$

where $d = \|f - s\|_{[a,b]}$, $\sigma \in \{-1, 1\}$, *and*

$$l = \#\{x_i\colon p+1 \leqslant i \leqslant p+q \text{ and } s \text{ is differentiable at } x_i\}.$$

Then s is a best approximation of f from $\mathscr{S}\mathscr{U}_k^0$.

Proof. Suppose $\tilde{s}$ is a better approximation of f than s; i.e., $\|f - \tilde{s}\| < d$. Then

$$(-1)^i \sigma(\tilde{s}(t_i) - s(t_i)) > 0, \qquad i = 0, ..., N,$$

It follows that $h := \tilde{s} - s$ has at least one zero z_i in each interval (t_i, t_{i+1}). Since $\|f - \tilde{s}\| < d$, h cannot be identically zero on (z_i, z_{i+1}). It follows that $D_+ h$ has at least $N-1$ strong sign changes on $J = (x_p, x_{p+q+1})$. We shall show that this is a contradiction.

First, note that h has at most $k+q-c$ knots in J, where c is the number of knots which are common to both s and $\tilde{s}$. Then h must be differentiable at at least $l-c$ of its knots. Since $D_+ h$ is piecewise in span($\mathscr{U}'$), it can have at most $m-2$ sign changes in each subinterval, and possibly one sign change at each knot where it has a jump. We conclude that the number of strong sign changes of $D_+ h$ in J satisfies

$$\begin{aligned} S^+(D_+ h) &\leqslant (q+1)(m-2) + k + q - c - (l - c) \\ &= (q+1)m + k - q - l - 2 = N - 2. \end{aligned}$$

This is the desired contradiction, and the proof is complete. ∎

Our next theorem gives a necessary condition for a best approximation.

THEOREM 3.3. *Assume that the space* $\mathscr{U}$ *satisfies the same assumptions as in Theorem* 3.2. *Let s be a best approximation of* $f \in C[a, b]$ *from* $\mathscr{S}\mathscr{U}_k^0$, *and suppose that the knots of s are* $a = x_0 < x_1 < \cdots < x_{r+1} = b$. *Let* $j_1 < \cdots < j_v$ *be the indices of the knots where* $D_+ s$ *has jumps, and let* $j_0 = 0$ *and* $j_{v+1} \equiv r+1$. *Then there must exist some* $0 \leqslant i \leqslant v$ *and an associated p, q such that* $j_i \leqslant p < p+q+1 \leqslant j_{i+1}$ *and*

$$A_{[x_p, x_{p+q+1}]}(f - s) \geqslant m(q+1) - 2q. \tag{3.2}$$

Proof. The idea of the proof is as follows: if s does not satisfy the necessary condition (3.2), we show how to construct another spline $\tilde{s} \in \mathscr{S}\mathscr{U}_k^0$ which is a better approximation. For each $i = 0, ..., v$, let

$$K_i = [x_{j_i}, x_{j_{i+1}}],$$

and let

$$\mathscr{S}_i = \mathscr{P}\mathscr{U}(\Delta_i) \cap C^1[K_i],$$

where $\Delta_i = \{x_{j_i}, x_{j_i+1}, ..., x_{j_{i+1}}\}$. This space has the *interlacing property* of [6], and thus by Theorem 2.2 of [8], a necessary condition for a function $g \in \mathcal{S}_i$ to be a best approximation of f on K_i is that there exist $x_{j_i} \leqslant x_p < x_{p+q+1} \leqslant x_{j_{i+1}}$ such that

$$A_{[x_p, x_{p+q+1}]}(f-g) \geqslant \dim \mathcal{S}_i = m(q+1) - 2q.$$

Our assumption asserts that $g_i = s|_{K_i}$ is *not* a best approximation of f on K_i from $\mathcal{S}_i$. Let $\tilde{g}_i$ be a best approximation, and define the sequence

$$g_{n,i} = g_i + (\tilde{g}_i - g_i)/n.$$

Clearly, $\|s - g_{n,i}\|_{K_i} \to 0$ as $n \to \infty$, and it is easy to see that for all n,

$$\|f - g_{n,i}\|_{K_i} < \|f - s\|_{K_i}. \tag{3.3}$$

Now in Lemma 3.4 below it is shown that the pieces $g_{n,i}$ can be joined together to construct a spline $\tilde{s} \in \mathcal{S}\mathcal{U}_k^0$. It is easy to check that this can be done so that the spline $\tilde{s}$ satisfies

$$\|f - \tilde{s}\|_{[a,b]} < \|f - s\|_{[a,b]}.$$

This completes the proof. ∎

The following lemma is used in the proof of Theorem 3.3 above, and will also be useful later.

LEMMA 3.4. *Suppose that $s \in \mathcal{S}\mathcal{U}^0(\Delta)$, where the functions in $\mathcal{U}$ lie in $C^1[a, b]$. Let $s_L = s|_{[a,\eta]}$ and $s_R = s|_{[\eta,b]}$, where $\eta \in \Delta$. In addition, suppose that*

$$s_L(\eta) = s_R(\eta) \qquad \text{and} \qquad D_- s_L(\eta) \neq D_+ s_R(\eta). \tag{3.4}$$

Let $\Delta_L = \Delta \cap [a, \eta]$ and $\Delta_R = \Delta \cap [\eta, b]$. Suppose that $s_{L,n} \in \mathcal{S}\mathcal{U}^0(\Delta_L)$ and $s_{R,n} \in \mathcal{S}\mathcal{U}^0(\Delta_R)$ are sequences of splines which converge uniformly to s_L and s_R on $J_L = [a, \eta]$ and $J_R = [\eta, b]$, respectively. We assume that $s_{L,n}$ is defined for all $x > \eta$ to be the same polynomial as in the interval ending at η. Similarly, we suppose that $s_{R,n}$ is defined for all $x < \eta$ to be the same polynomial as in the interval beginning at η. Then for all sufficiently large n, there exists a point ξ_n such that

$$s_{L,n}(\xi_n) = s_{R,n}(\xi_n).$$

Moreover, $\xi_n \to \eta$ as $n \to \infty$.

Proof. Without loss of generality, we may suppose that $D_- s_L(\eta) - D_+ s_R(\eta) = 2\varepsilon > 0$. Then for all sufficiently small δ, we have

$$\frac{s_L(\eta+\delta) - s_L(\eta)}{\delta} > \frac{s_R(\eta+\delta) - s_R(\eta)}{\delta} + \varepsilon.$$

Since $s_L(\eta) = s_R(\eta)$, it follows that

$$s_L(\eta+\delta) - s_R(\eta+\delta) > \varepsilon\delta.$$

A similar argument shows that

$$s_L(\eta-\delta) - s_R(\eta-\delta) < -\varepsilon\delta.$$

The function $s_L - s_R$ changes sign at η. It then follows that for all n sufficiently large, the function $s_{L,n} - s_{R,n}$ must also change sign at some point in the interval $(\eta-\delta, \eta+\delta)$. We take ξ_n to be any such point. Since δ was arbitrarily small, we can make sure that ξ_n is arbitrarily close to η. ∎

We have not obtained a complete characterization of best approximations from $\mathscr{S}\mathscr{U}_k^0$. Indeed, the sufficient conditions of Theorem 3.2 do not coincide with the necessary conditions of Theorem 3.3. The following example shows a typical case.

EXAMPLE 3.5. Let f be the piecewise linear function which interpolates the values $\{1, -1, 1, 1, 2, 1, 4\}$ at the points $\{-3, -2, ..., 3\}$, and consider approximation from the space $\mathscr{S}\mathscr{U}_1^0$ with $u_1(x) = 1$ and $u_2(x) = x$.

Discussion. The unique best approximation in this case is $s(x) = x_+$. By Theorem 3.2, sufficient conditions for a best approximation are that $f - s$ alternates at least three times on either $[-3, 0]$ or $[0, 3]$, or at least four times on $[-3, 3]$. Theorem 3.3 asserts that a necessary condition for a best approximation is that $f - s$ alternates at least two times on one of the intervals $[-3, 0]$ or $[0, 3]$.

In the remainder of this section we assume that $f \in \mathscr{K}(\mathscr{U})$, where $\mathscr{K}(\mathscr{U})$ is the convexity cone (2.3) associated with $\mathscr{U}$. As we saw in Section 2, in the case where the dimensionality m of $\mathscr{U}$ is even, approximation from $\mathscr{P}\mathscr{U}_k$ is the same as approximation from $\mathscr{S}\mathscr{U}_k^0$. Thus, we now restrict our attention to the case where m is odd. The following theorem shows that in this case approximation from the two spaces is indeed different.

THEOREM 3.6. *Suppose $\mathscr{U}$ is a normed Tchebycheff system of at least $m \geq 3$ functions with m odd. Then for every $f \in \mathscr{K}(\mathscr{U})$,*

$$d(f, \mathscr{S}\mathscr{U}_k^0) > d(f, \mathscr{P}\mathscr{U}_k). \tag{3.5}$$

Proof. Let s be the best approximation of f from $\mathscr{P}\mathscr{U}_k$. By Theorem 2.7, s has exactly k knots $a=x_0<x_1<\cdots<x_{k+1}=b$, and it alternates exactly m times on each subinterval $[x_i, x_{i+1}]$ with both endpoints being peak points. Since m is odd, in fact we have

$$(f-s)(x_{i^-}) = -(f-s)(x_{i^+}) = \sigma d(f, \mathscr{P}\mathscr{U}_k), \qquad i=1, ..., k,$$

where $\sigma \in \{-1, 1\}$. Let s^0 be the best approximation of f from $\mathscr{S}\mathscr{U}_k^0$, and suppose that it has knots $a=y_0<\cdots<y_{r+1}=b$ with $r \leqslant k$. Since $\mathscr{S}\mathscr{U}_k^0 \subseteq \mathscr{P}\mathscr{U}_k$, we have $\|f-s\|_{[a,b]} \leqslant \|f-s^0\|_{[a,b]}$.

Now if the knot sets of s and s^0 are different, then for some i, j, we have $[x_i, x_{i+1}] \subset [y_j, y_{j+1}]$. Then using Lemma 2.4, it follows that

$$\begin{aligned} d(f, \mathscr{S}\mathscr{U}_k^0) &\geqslant \|f-s\|_{[a,b]} \geqslant \|f-s\|_{[y_j, y_{j+1}]} \\ &> \|f-s\|_{[x_i, x_{i+1}]} = d(f, \mathscr{P}\mathscr{U}_k). \end{aligned}$$

To complete the proof, we must consider the case where the knot sets of s and s^0 are identical. Suppose $\|f-s\|_{[a,b]} = \|f-s^0\|_{[a,b]}$. Then on each interval $[x_i, x_{i+1})$, the alternation properties of s imply that $s \equiv s^0$. But this is impossible since s^0 is continuous while s is not. ∎

In the remainder of this section we restrict $\mathscr{U}$ and f even further. Suppose $\mathscr{U}$ is a set of m functions as in Theorem 3.2 with m odd. Associated with $\mathscr{U}$, define the cone of functions

$$\mathscr{K}^1(\mathscr{U}) = \{f \in C^1[a,b] \cap \mathscr{K}(\mathscr{U}): f' \in \mathscr{K}(\mathscr{U}')\},$$

where $\mathscr{U}'$ is as in (3.1). First we present a lemma concerning best approximation of functions in $\mathscr{K}^1(\mathscr{U})$ by splines in $\mathscr{S}\mathscr{U}^0(\Delta)$, where Δ is a given set of knots $a=x_0<x_1<\cdots<x_{k+1}=b$. As shown in [6],

$$\dim \mathscr{S}\mathscr{U}^0(\Delta) = N = (k+1)(m-1)+1. \tag{3.6}$$

This space has the interlacing property discussed in [6].

LEMMA 3.7. *Suppose $\mathscr{U}$ is a set of m functions as in Theorem 3.2 with m odd, and suppose Δ is a given partition of $[a,b]$ with k knots. Suppose $f \in \mathscr{K}^1(\mathscr{U})$ and $s \in \mathscr{S}\mathscr{U}^0(\Delta)$. Then*

$$Z_{[a,b]}(f-s) \leqslant N, \tag{3.7}$$

where Z counts the number of zeros, and N is defined in (3.6). *Moreover, if $Z_{[a,b]}(f-s)=N$, then the set of zeros $\{t_i\}_1^N$ of $f-s$ is* poised *with respect to $\mathscr{S}\mathscr{U}^0(\Delta)$; i.e., if $\{s_1, ..., s_N\}$ is any basis for $\mathscr{S}\mathscr{U}^0(\Delta)$, then*

$$D\begin{pmatrix} t_1, ..., t_N \\ s_1, ..., s_N \end{pmatrix} = \det(s_j(t_i))_{i,j=1}^N \neq 0. \tag{3.8}$$

Proof. Let $g = f - s$ and $g_i = g|_{[x_i, x_{i+1})}$ for $i = 0, ..., k$. Since $f \in \mathscr{K}^1(\mathscr{U})$, g_i' has at most $m-1$ distinct zeros in $[x_i, x_{i+1}]$. If g_i' has exactly $m-1$ distinct zeros $z_{i,1} < \cdots < z_{i,m-1}$, then

$$g_i'(x) = D\begin{pmatrix} z_{i,1}, ..., z_{i,m-1}, x \\ u_2', ..., u_m', f' \end{pmatrix} \Big/ D\begin{pmatrix} z_{i,1}, ..., z_{i,m-1} \\ u_2', ..., u_m' \end{pmatrix}.$$

This implies that

$$g_i'(x) > 0 \qquad \text{for} \quad x_i < x < z_{i,1} \quad \text{and} \quad z_{i,m-1} < x < x_{i+1}.$$

It follows that $g_i'(x_{i-})\, g_i'(x_{i+}) < 0$ can happen only if g_i' has at most $m-2$ zeros in at least one of the intervals $[x_{i-1}, x_i]$ or $[x_i, x_{i+1}]$. Since m is odd,

$$Z_{[a,b]}(D_+ g) \leqslant N - 1 = (k+1)(m-1),$$

where Z also counts jump zeros. Now Rolle's theorem implies (3.7).

Suppose now that $Z_{[a,b]}(g) = N$. In particular, suppose that the zeros of g are $z_1 < \cdots < z_N$. We claim that

$$z_{N - n_{i,k+1}} < x_i < z_{n_{0,i}+1}, \qquad i = 1, ..., k, \tag{3.9}$$

where

$$n_{i,j} = \dim \mathscr{S}\mathscr{U}^0(\Delta)|_{[x_i, x_j]} = (j-i)(m-1) + 1. \tag{3.10}$$

Indeed, by the above arguments,

$$Z_{[x_0, x_i]}(g) \leqslant i(m-1) + 1$$

and

$$Z_{[x_i, x_{k+1}]}(g) \leqslant (k+1-i)(m-1) + 1.$$

These inequalities imply

$$z_{i(m-1)+2} > x_i \qquad \text{and} \qquad z_{N-(k+1-i)(m-1)-1} < x_i \tag{3.11}$$

for each $i = 1, ..., k$. These are precisely the inequalities in (3.9) since $n_{0,i} = i(m-1)+1$ and $n_{i,k+1} = (k+1-i)(m-1)+1$. Since $\mathscr{S}\mathscr{U}^0(\Delta)$ has the interlacing property, (3.9) implies (3.8). ∎

W can now establish several useful properties of best approximants of functions f in $\mathscr{K}^1(\mathscr{U})$.

THEOREM 3.8. *Let $\mathscr{U}$ be as in Theorem 3.2 with m odd, and suppose*

$f \in \mathcal{K}^1(\mathcal{U})$. Then any best approximation s of f from $\mathcal{S}\mathcal{U}_k^0$ must lie in $\mathcal{S}\mathcal{U}_k^1 \backslash \mathcal{S}\mathcal{U}_{k-1}^1$. Moreover,

$$A_{[x_i, x_j]}(f-s) < n_{i,j}, \qquad \text{all } i, j \text{ such that } [x_i, x_j] \neq [a, b] \quad (3.12)$$

$$A_{[a,b]}(f-s) = N. \quad (3.13)$$

Here $n_{i,j}$ and N are defined in (3.10) *and* (3.6), *respectively.*

Proof. Let s be a best approximation of f from $\mathcal{S}\mathcal{U}_k^0$, and suppose that the knots of s are $\Delta = \{a = x_0 < x_1 < \cdots < x_{r+1} = b\}$. We prove first that $r = k$. Suppose $r < k$. Then we can insert a knot x_{r+2} and renumber the knots as $\tilde{\Delta} = \{a = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_{r+2} = b\}$. This knot can be chosen so that it lies in the interval whose endpoints are the last two alternating extreme points of $f - s$. But then s cannot be a best approximation of f from the space $\mathcal{S}\mathcal{U}^0(\tilde{\Delta})$, and hence also not from the space $\mathcal{S}\mathcal{U}_k^0$, since the characterization Theorem 2.2 for approximation by generalized splines using fixed knots given in [8] is violated. This contradiction establishes that $r = k$.

Now applying the characterization Theorem 2.2 of [8] to the space $\mathcal{S}\mathcal{U}^0(\Delta)$, we conclude that there exist some $0 \leqslant p, q \leqslant r + 1$ such that

$$A_{[x_p, x_q]}(f-s) = n_{p,q} \quad (3.14)$$

but

$$A_{[x_i, x_j]}(f-s) < n_{i,j} \quad (3.15)$$

for every choice of i, j with $[x_i, x_j]$ a proper subset of $[x_p, x_q]$.

We claim that both x_p and x_q are peak points of $f - s$, and that both $D_+(f-s)(x_p)$ and $D_-(f-s)(x_q)$ are nonzero. To prove this, suppose that x_p is not a peak point or that $D_+(f-s)(x_p) = 0$. Then $D_+(f-s)$ would have zeros or sign changes at each of the first $n_{p,q}$ peak points of $f - s$. But as shown in the proof of Lemma 3.7, $D_+(f-s)$ can have at most $n_{p,q} - 1$ zeros in $[x_p, x_q]$. This contradiction establishes the assertion that x_i is a peak point of $f - s$ and that $D_+(f-s)(x_p) \neq 0$. A similar argument takes care of x_q.

We claim now that (3.14), (3.15) actually hold for $p = 0$ and $q = r + 1$. Suppose that this is not the case. In particular, suppose that $0 \leqslant p_1 < q_1 < p_2 < \cdots < p_\rho < q_\rho \leqslant r + 1$ are such that

$$A_{[x_{p_j}, x_{q_j}]}(f-s) \geqslant n_{p_j, q_j}, \qquad j = 1, ..., \rho,$$

while (3.15) holds for all i, j with $[x_i, x_j]$ a proper subset of $[x_{p_j}, x_{q_j}]$. Moreover, suppose that

$$A_{[x_\mu, x_\nu]}(f-s) < n_{\mu,\nu}$$

for all subintervals $[x_\mu, x_\nu] \subset [x_{q_j}, x_{p_{j+1}}]$ for all $j = 0, ..., \rho$, where $q_0 = 0$ and $p_{\rho+1} = r+1$. All of the endpoints of these intervals are peak points of $f - s$ with nonzero derivatives. Now on each of the intervals $[x_{q_j}, x_{p_{j+1}}]$, s is not a best approximation with respect to $\mathscr{S}\mathscr{U}^0(\Delta)|_{[x_{q_j}, x_{p_{j+1}}]}$, and hence we can find a perturbation s_j which is better. Proceeding just as in the proof of Lemma 3.4, these pieces can be joined together with the pieces $s|_{[x_{p_j}, x_{q_j}]}$ to get a spline $\tilde{s}$ with

$$\|f - \tilde{s}\| = d(f, \mathscr{S}\mathscr{U}_k^0) \tag{3.16}$$

for which (3.15) holds for all i, j. But this is a contradiction since (3.16) implies that $\tilde{s}$ is a best approximation of f among all splines with the same fixed knots, while the lack of an interval satisfying (3.14) asserts that $\tilde{s}$ cannot be a best approximation of f from this space. This completes the proof of (3.12), (3.13).

It remains to show that s belongs to $C^1[a, b]$. Suppose s is not differentiable at the knot x_i. By what we have already established, we know that $f - s$ alternates N times on $[a, b]$. Let the set of alternating peak points be denoted by T. By (3.15), at most $n_{0,i}$ of these points fall in $[x_0, x_i]$, while at most $n_{i,r+1}$ of them all in $[x_i, x_{r+1}]$. Since $\# T = N + 1 = n_{0,i} + n_{i,r+1}$, it follows that $x_i \notin T$. Now $f - s$ does not alternate enough times to make s a best approximation (with fixed knots) on either of the intervals $[a, x_i]$ or $[x_i, b]$. But then we can perturb s on both of these intervals and then use Lemma 3.4 to join them together with a new knot (replacing x_i) to show that s could not have been a best approximation of f. This contradiction establishes the differentiability of s, and the theorem is proved. ∎

THEOREM 3.9. *Let $\mathscr{U}$ be a set of functions as in Theorem 3.8. Suppose f and g are two functions such that both $f + g$ and $f - g$ belong to $\mathscr{K}^1(\mathscr{U})$. Then*

$$d(g, \mathscr{S}\mathscr{U}_k^0) \leqslant d(f, \mathscr{S}\mathscr{U}_k^0). \tag{3.17}$$

Proof. The assumption that both $f + g$ and $f - g$ lie in $\mathscr{K}^1(\mathscr{U})$ implies that f itself lies in $\mathscr{K}^1(\mathscr{U})$. Now Theorem 3.8 asserts that if s is a best approximation of f from $\mathscr{S}\mathscr{U}_k^0$ with knots $\Delta = \{a = x_0 < x_1 < \cdots < x_{k+1} = b\}$, then $f - s$ alternates exactly $N = \dim \mathscr{S}\mathscr{U}^0(\Delta)$ times in $[a, b]$ to a height of $d = d(f, \mathscr{S}\mathscr{U}_k^0)$.

Let $s_1, ..., s_N$ be a basis for $\mathscr{S}\mathscr{U}^0(\Delta)$. We claim now that both $\{s_1, ..., s_N, f + g\}$ and $\{s_1, ..., s_N, f - g\}$ are weak-Tchebycheff systems on $[a, b]$. To establish this, let $h = (f + g) - v$ with $v \in \mathscr{S}\mathscr{U}^0(\Delta)$. Lemma 3.7 now applies to assert that h has at most N zeros on $[a, b]$, and the assertion with $f + g$ immediately follows. The assertion with $f - g$ can be established in exactly the same way.

Now applying Theorem 2 of [9], we get

$$d(g, \mathcal{S}\mathcal{U}_k^0) \leqslant \inf_{s \in \mathcal{S}\mathcal{U}^0(\Delta)} \|g - s\| \leqslant \inf_{s \in \mathcal{S}\mathcal{U}^0(\Delta)} \|f - s\| = d(f, \mathcal{S}\mathcal{U}_k^0). \quad \blacksquare$$

Theorem 3.9 is a generalization of well-known comparison theorems. In particular, if $\mathcal{U} = \{1, x, ..., x^{m-1}\}$, and $f^{(m)}(x) > |g^{(m)}(x)|$ for all $x \in [a, b]$, then the hypotheses of Theorem 3.9 hold.

4. Approximation from the Space $\mathcal{S}\mathcal{U}_k^l$ for $l \geqslant 1$

In this section we show that for $l \geqslant 1$, the space $\mathcal{S}\mathcal{U}_k^l$ is not useful for approximation, even when $\mathcal{U}$ is very nice.

Example 4.1. Let $l \geqslant 1$, and let $g(x) = x_+^l$. Let f be a function in $C^{l-1}[-1, 1]$ such that $\|f - g\| \leqslant 1$, $f(0) = -1$ and

$$f(-ih) = (-1)^{i-1}, \qquad i = 1, ..., 5l$$

$$f(ih) = (-1)^{i-1} + (ih)^l, \qquad i = 1, ..., 5l,$$

where $h = 1/(5l)$. Let $\mathcal{U}$ be the space of polynomials of degree $l + 1$, and consider approximating f by $\mathcal{S}\mathcal{U}_2^l$.

Discussion. Clearly, for each $n > 0$, the spline $s_n(x) = n/(l+1) [x_+^{l+1} - (x - 1/n)_+^{l+1}]$ belongs to $\mathcal{S}\mathcal{U}_2^l$. Moreover, $\|f - s_n\| \to 1$ as $n \to \infty$, and thus $d(f, \mathcal{S}\mathcal{U}_2^l) \leqslant 1$. Now suppose that s is a best approximation of f from $\mathcal{S}\mathcal{U}_2^l$. Then we must have

$$(-1)^{i-1} s(-ih) > 0$$

and

$$(-1)^{i-1} s(ih) + (ih)^l > 0, \qquad i = 1, ..., 5l.$$

This implies that the function $s - g$ must vanish in each of the intervals $L_i := (-(i+1)h, -ih)$ and $R_i := (ih, (i+1)h)$ for $i = 1, ..., 5l - 1$. It follows that $s \equiv g$ on some interval $[a, b]$ with $b < 0$, and on some interval $[c, d]$ with $0 < c$. But then s must have the form

$$s(x) = e[(x - \alpha_1)_+^{l+1} - (x - \alpha_2)_+^{l+1}]$$

while it must also satisfy the condition $s(x) = x^l$ for $x > \alpha_2$. For $l \geqslant 2$ it is easy to see that this is impossible. On the other hand, for $l = 1$, the condition requires that $\alpha_2 = -\alpha_1 = \alpha > 0$ and $e > 0$. But then $s(0) > 0$, and so $\|f - s\| > 1$ and s cannot be a best approximation since as noted above, $d(f, \mathcal{S}\mathcal{U}_2^l) \leqslant 1$. This contradiction completes the proof.

5. Polynomial Splines with Free Knots

In this section we apply the idea inherent in Lemma 3.4 to obtain a new result for the set $\mathscr{S}_{m,k}$ of polynomial splines of degree m with at most k free knots, counting multiplicities (cf. [10]). Various necessary conditions and sufficient conditions for a spline $s \in \mathscr{S}_{m,k}$ to be a best approximation of a given $f \in C[a, b]$ can be found in [1]. For example, it was shown in [1] that if $s \in C[a, b]$, then it is a best approximation of f only if there exists some p, q such that

$$A_{[x_p, x_{p+q+1}]}(f-s) \geqslant m + \sum_{i=p+1}^{p+q} m_i + \tilde{q} + 1,$$

where $m_1, ..., m_r$ are the multiplicities of the knots of s, and where $\tilde{q}$ is the number of knots in (x_p, x_{p+q+1}) with multiplicity at most $m-1$. The following theorem is an improvement of this result.

Theorem 5.1. *Suppose $s \in \mathscr{S}_{m,k}$ is a best approximation of $f \in C[a, b]$ with knots $a = x_0 < x_1 < \cdots < x_{r+1} = b$. Then there must exist some p, q such that $s \in C^1[x_p, x_{p+q+1}]$ and*

$$A_{[x_p, x_{p+q+1}]}(f-s) \geqslant m + \sum_{i=p+1}^{p+q} m_i + q + 1, \tag{5.1}$$

where for each $i = 1, ..., r$, the integer m_i represents the multiplicity of the knot x_i.

Proof. The analysis divides into two cases.

Case 1 ($m_i \leqslant m$ for all $i = 1, ..., r$). In this case $s \in C[a, b]$. Let E be the set of extreme points of $f - s$; i.e., $E = \{t \in [a, b]: |f(t) - s(t)| = \|f - s\|\}$. Let $1 \leqslant v_1 < \cdots < v_l \leqslant r$ be such that x_{v_j} has multiplicity m, and let $x_{v_0} = a$ and $x_{v_{l+1}} = b$.

Case 1A. Suppose there exists $0 \leqslant \mu \leqslant l$ such that 0 is a best approximation of $f - s$ on $[x_{v_\mu}, x_{v_{\mu+1}}]$ with respect to the space $\mathscr{S}_\mu$ of polynomial splines on $[x_{v_\mu}, x_{v_{\mu+1}}]$ of degree m with knots at $x_{v_\mu+1}, ..., x_{v_{\mu+1}-1}$ of multiplicities $m_{v_\mu+1} + 1, ..., m_{v_{\mu+1}-1} + 1$. Since $m_i \leqslant m - 1$ for $i = v_\mu + 1, ..., v_{\mu+1} - 1$, it follows that $\mathscr{S}_\mu \subset C^1[x_{v_\mu}, x_{v_{\mu+1}}]$. Then using the classical characterization theorem for fixed knots (cf. [5]), there must exist some p, q with $v_\mu \leqslant p < p + q + 1 \leqslant v_{\mu+1}$ such that

$$A_{[x_p, x_{p+q+1}]}(f-s) \geqslant m + \sum_{j=p+1}^{p+q} m_j + q + 1.$$

Case 1B. Suppose that for all $0 \leqslant \mu \leqslant l$ the function 0 is *not* a best approximation of $f-s$ on $[x_{v_\mu}, x_{v_{\mu+1}}]$ with respect to the space $\mathscr{S}_\mu$ defined above. Then by the tangent method in [4] it follows that for each $0 \leqslant \mu \leqslant l$, there exists a sequence of splines $s_{\mu,n}$ on $[x_{v_\mu}, x_{v_{\mu+1}}]$ with the same number of multiple knots as s such that

$$\|f - s_{\mu,n}\|_{[x_{v_\mu}, x_{v_{\mu+1}}]} < \|f - s\|_{[x_{v_\mu}, x_{v_{\mu+1}}]}$$

for all n and

$$\lim_{n \to \infty} \|s - s_{\mu,n}\| = 0.$$

Now arguing as in Lemma 3.4 (cf. also the proof of Theorem 3.8), it follows that we can construct a spline $\tilde{s} \in \mathscr{S}_{m,k}$ with $\|f-\tilde{s}\| < \|f-s\|$. This contradicts the fact that s is a best approximation of f from $\mathscr{S}_{m,k}$.

Case 2 ($m_i = m+1$ for some $1 \leqslant i \leqslant r$). Let $\{j_1 < \cdots < j_l\}$ be such that the corresponding knots x_{j_i} have multiplicity $m+1$. Let $x_{j_0} = a$ and $x_{j_{l+1}} = b$. Suppose there is some $1 \leqslant v \leqslant l$ such that s is a best approximation of f on $[x_{j_v}, x_{j_{v+1}}]$ with respect to the set of splines with the same number of multiple knots as s in this interval. Then the assertion follows as in Case 1. Otherwise, for each $j = 0, ..., l$, there exists a spline $\tilde{s}_j$ with the same number of multiple knots as s on the interval $[x_{j_v}, x_{j_{v+1}}]$ which is better than s. But then

$$\tilde{s}(t) = \begin{cases} \tilde{s}_0(t), & x_{j_0} \leqslant t < x_{j_1} \\ ..., & \\ \tilde{s}_l(t), & x_{j_l} \leqslant t \leqslant x_{j_{l+1}} \end{cases}$$

would be a better approximation of f in $\mathscr{S}_{m,k}$, which is a contradiction. ∎

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